The $2m \leq r$ property of spherically symmetric static spacetimes

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Abstract

We prove that all spherically symmetric static spacetimes which are both regular at $r = 0$ and satisfying the single energy condition $\rho + p_r + 2p_t \geq 0$ cannot contain any black hole region (equivalently, they must satisfy $2m/r \leq 1$ everywhere). This result holds even when the spacetime is allowed to contain a finite number of matching hypersurfaces. This theorem generalizes a result by Baumgarte and Rendall when the matter contents of the space-time is a perfect fluid and also complements their results in the general non-isotropic case.

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In a recent paper, Baumgarte and Rendall [1] have proven that all spherically symmetric static perfect-fluid solutions of the Einstein field equations with a regular centre cannot enter into a black hole region (up to where the pressure vanishes) provided that the energy-density ρ is continuous and non-negative and the central isotropic pressure is finite and positive. The method for obtaining their result was the analysis of the Tolman-Oppenheimer-Volkoff equation (we assume $8\pi G = c = 1$ throughout)

$$\frac{dp}{dr} = -\frac{(\rho + p)\,(2m(r) + pr^3)}{2r^2\,(1 - 2m(r)/r)}$$

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where ρ and p are the energy density and pressure of the perfect fluid and $m(r)$ is the so-called mass function defined by

$$m(r) \equiv \frac{1}{2}\int_0^r \rho(v)v^2 dv. \tag{1}$$

Here r stands for the usual Schwarzschild area coordinate (see below). Their main result was that, in the region where p remains non-negative (which is the interior of the autogravitating fluid), we have

$$\frac{2m}{r} < 1, \qquad \text{in the region where} \quad p(r) \geq 0, \tag{2}$$

which clearly implies that the spacetime cannot contain any black hole region (there are no trapped 2-spheres). This theorem was a generalization of previous results [2] which assumed also the existence of a barotropic equation of state $p = p(\rho)$ satisfying $dp/d\rho > 0$.

In the same Ref.[1], the impossibility of entering into a black hole region was also shown in the case that the fluid has anisotropic pressures. In this case, they proved that condition (2) holds up to where the radial pressure vanishes (again the interior of the autogravitating fluid) under the similar assumptions of continuous and non-negative energy density, positive and finite central radial pressure and a C^1 tangential pressure. The additional condition they assumed of tangential pressure being coincident with the radial pressure at $r = 0$ is, in fact, a direct consequence of the regularity of the spacetime at $r = 0$.

The aim of this paper is to present a new theorem on spherically symmetric static spacetimes which is a strict generalization of the theorem by Baumgarte and Rendall in the perfect-fluid case and that largely complements their results in the completely general case. In fact, we shall prove by using very simple arguments that an arbitrary spherically symmetric static spacetime satisfying the single condition

$$\rho + p_r + 2p_t \geq 0, \tag{3}$$

(where ρ, p_r and p_t are defined below) must necessarily fulfil

$$\frac{2m}{r} \leq 1 \qquad \forall r > 0.$$

The only differentiability requirements we impose are that the matching conditions are fulfilled everywhere (equivalently, there are no surface layers) and that the spacetime metric is C^2 between any two consecutive matching hypersurfaces. These requirements do not even imply the continuity of the energy density ρ, nor the continuity of the tangential pressure p_t.

In order to prove the theorem, let us choose coordinates $\{t, R, \theta, \phi\}$ in which the metric takes the form

$$ds^2 = -F^2(R)dt^2 + dR^2 + r^2(R)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4}$$

where $r(R)$ is the usual Schwarzschild radius. These coordinates are always well-defined in at least a neighbourhood of $r = 0$ *as long as* the spacetime is regular there. The regularity of the spacetime on a centre of symmetry $R = 0$ (we can fix an additive constant in R in order to set $r(0) = 0$, that is to say, we can choose R such that a regular centre of symmetry is at $R = 0$) is satisfied if and only if the metric functions $r(R)$ and $F(R)$ have the following asymptotic behaviour

$$r(R \to 0) \rightsquigarrow R - \frac{m_0}{3}R^3, \quad F(R \to 0) \rightsquigarrow F_0 + F_1\, R^2, \quad F_0 > 0 \tag{5}$$

where m_0, F_0 and F_1 are constants and F_0 is strictly positive. These conditions will be assumed from now on. This is one of the main assumptions in our work, as otherwise there is no guarantee that the coordinates of (4) exist.

The energy-momentum tensor of this spacetime can be calculated from its Einstein tensor via the Einstein equations. Using the static velocity vector

$$\vec{u} = \frac{1}{F}\frac{\partial}{\partial t}$$

to decompose the energy-momentum tensor, we find that the energy-density ρ, radial pressure (or tension) p_r and tangential pressure (or tension) p_t with respect to $\vec{u}$ are, respectively

$$\rho = \frac{1}{r^2}\left(1 - r'^2 - 2rr''\right), \quad p_r = \frac{1}{r^2}\left(-1 + r'^2 + 2rr'\frac{F'}{F}\right), \quad p_t = \frac{F''}{F} + \frac{F'}{F}\frac{r'}{r} + \frac{r''}{r}$$

where the prime denotes derivative with respect to R.

As is well-known the mass function m is invariantly defined for every spherically symmetric spacetime [3], and in our case is given by the expression

$$2m \equiv r\left(1 - r'^2\right), \tag{6}$$

from where one can immediately obtain the formula (1) previously given. In consequence, we have that, *as long as* the coordinate system we have chosen is appropriate, the mass function is bounded to satisfy

$$\frac{2m}{r} \leq 1,$$

the equality holding *only* at points with $r' = 0$, if any. Therefore the spacetime does not contain closed trapped surfaces (which are the indication of the existence of a black

hole region) in the domain where the chosen coordinates are valid. Thus, proving that the spacetime cannot contain a back hole region under the assumption $\rho + p_r + 2p_t \geq 0$ amounts to proving that the coordinates $\{t, R\}$ in which the metric (4) is written cover the whole manifold and that this manifold is inextendible.

As is obvious, from expression (4) follows that the coordinates $\{t, R\}$ will fail describing the spacetime manifold if and only if the metric function $F(R)$ vanishes somewhere (if the function F diverges for some R, this is either a singularity of the spacetime or we have $r = 0$ as can be easily checked from expression (7) below when considered as a second order linear differential equation for F). It is also a simple matter of checking that if $F(R)$ is everywhere positive then the manifold is inextendible (all causal curves are inextendible). As we explained before, the vanishing of $r(R)$ at $R = 0$ is just the indication that we are at the centre of symmetry of the spacetime. The function $r(R)$ may certainly vanish again at some other value of $R > 0$, but this is either *another* regular centre of symmetry or a curvature singularity of the spacetime. In both cases, the coordinate system covers the whole manifold and the inequality $2m \leq r$ will hold everywhere.

From the above we see that, in order to prove our result, we only have to show that the function $F(R)$ cannot vanish anywhere under the assumptions that the matter content satisfies the condition (3) and that there is *a* regular centre of symmetry $R = 0$. The main assumption of our theorem, namely the condition (3), reads explicitly

$$\frac{F''}{F} + 2\frac{F'}{F}\frac{r'}{r} = \frac{1}{2}\left(\rho + p_r + 2p_t\right) \geq 0. \tag{7}$$

Consequently, the lefthand side of this relation must be a non-negative function. This condition can be rewritten as

$$\left(F'r^2\right)' = \frac{1}{2}\left(\rho + p_r + 2p_t\right) r^2 F. \tag{8}$$

or, equivalently,

$$F'(R) = \frac{1}{2r^2}\int_0^R \left(\rho + p_r + 2p_t\right) r^2 F d\tilde{R}. \tag{9}$$

It is clear that this equation, together with (3) and (5), implies that F is positive everywhere. This follows immediately from the fact that $F(0) = F_0 > 0$, so that there is a non-empty interval $[0, R_1]$ in which F is strictly positive, together with (9) and (3) which imply then that $F'(R)$ is non-negative in this interval. Therefore, F is a non-decreasing function in $[0, R_1]$ and thus $F(R_1) \geq F_0 > 0$. Proceeding in this manner (starting now at R_1) we see that F is a non-decreasing function everywhere and in fact we have $F(R) \geq F_0 > 0$ for all possible R.

Now, this proof implicitly assumes that both functions $F(R)$ and $r(R)$ are piecewise C^2 (so that the expressions for ρ, p_r and p_t make sense) and that the function F is

C^1 everywhere (otherwise we are not allowed to integrate (8) to give simply (9)). Our aim now is to show that this differentiability requirements are fulfilled if the spacetime under consideration contains a finite number of matching hypersurfaces. Let us here emphasize that, by allowing for the existence of those matching hypersurfaces, we are including in our treatment spacetimes representing not only the interior of a star but also its exterior, which can be either empty or filled with some exterior field such as, for example, an electromagnetic field. Notice that in all the previous theorems (e.g. in [1]) only the region up to the limit surface of the star was considered.

By assuming the absence of surface layers, the junction conditions reduce to the equality of both the first and the second fundamental forms calculated from the two sides of the matching hypersurface (see e.g. [4][5]). Let us remark here that, in general, these conditions do not imply the continuity of the first derivatives of the metric coefficients in a prefixed coordinate system (see, for instance, [4][5]). Nevertheless, a straightforward calculation shows that, in our case, the junction conditions on the matching hypersurfaces $R = \bar{R} = \text{const.}$ reduce simply to

$$\begin{aligned} F|_{R\to\bar{R}^+} &= F|_{R\to\bar{R}^-}\,, & r|_{R\to\bar{R}^+} &= r|_{R\to\bar{R}^-}\,, \\ F'|_{R\to\bar{R}^+} &= F'|_{R\to\bar{R}^-}\,, & r'|_{R\to\bar{R}^+} &= r'|_{R\to\bar{R}^-}\,, \end{aligned}$$

so that both functions $F(R)$ and $r(R)$ have continuous derivatives everywhere and the result is established. We may notice by the way that this result means that the coordinate system $\{t, R, \theta, \phi\}$ of (4) is *admissible* in the sense of Lichnerowicz [5][6]. We have thus established the following theorem.

Theorem 1 *In an arbitrary static and spherically symmetric spacetime which is piecewise C^2 and satisfies the matching conditions on a finite number of hypersurfaces, if the spacetime is regular at a centre of symmetry and satisfies $\rho + p_r + 2p_t \geq 0$, then $2m \leq r$ everywhere.*

Let us remark that this result has been proved irrespective of the matter content of the spacetime (provided that the condition $\rho + p_r + 2p_t \geq 0$ holds everywhere). Thus, the energy-momentum tensor can certainly represent a real static fluid, but also more complicated situations in which matter and/or additional fields (typically electromagnetic fields) are present. The case with empty regions is also covered by our treatment.

On the other hand, let us notice that the strict inequality $2m/r < 1$ holds everywhere except at points where $r' = 0$, as we already remarked after Eq.(6). However, at those points, the expression for p_r becomes

$$p_r|_{r'=0} = -\frac{1}{r^2} < 0.$$

Thus, the equality $2m = r$ is forbidden as long as p_r remains positive. In other words, the strict inequality $2m < r$ of Ref.[1] is fully recovered because in [1] this was

shown only up to where p_r vanishes. It may be interesting to remark that the possible hypersurfaces with $r' = 0$ (or equivalently $2m = r$) are completely ordinary *timelike* hypersurfaces if F does not vanish there (and F does not vanish if (3) and (5) hold, as we have seen). Therefore, the possible hypersurfaces $2m = r$ which appear in our treatment do not have anything to do with horizons or other null hypersurfaces, and the Killing vector $\partial/\partial t$ is timelike everywhere if condition (3) holds.

To end this letter, let us briefly comment on the main condition (3) we have assumed in the theorem. This condition is implied by (but is much less restrictive than) the strong energy condition (see, e.g., [7]). For the spherically symmetric static case we are dealing with, the strong energy condition is equivalent to

$$\rho + p_r \geq 0, \qquad \rho + p_t \geq 0, \qquad \rho + p_r + 2p_t \geq 0.$$

Thus, our condition (3) is *just* one of the consequences of the strong energy condition.

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